Quasicrossing distribution as a signature of the onset of chaos in the SU(3) nuclear model

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(Received 9 November 1992)

The transition from the regular to the chaotic regime in a simple model has been studied by two different methods: the stability matrix and the quasicrossing distribution. Good agreement between the two methods was obtained.

PACS number(s): 05.45.+b, 03.65.Sq, 24.60.−k

In the past few years many authors, working in different fields, have shown great interest in the so-called "*quantum chaos*" or "*quantum chaology,*" i.e., the signature in quantal systems of the chaotic properties of the corresponding ($\hbar \to 0$) semiclassical Hamiltonian [1,2].

In this spirit we compare in this paper the results obtained by two different approaches: a classical one based on the stability matrix and two quantal criteria using the distribution of quasicrossings and the Δ^2 statistics.

The model used is the three-level Lipkin-Meshow-Glick one [3,4], whose Hamiltonian is

$$H=\sum_{k=0}^{2} \varepsilon_k G_{kk} - \frac{V}{2} \sum_{k,l=0}^{2} G_{kl}^2 , \tag{1}$$

where

$$G_{kl}=\sum_{m=1}^{M} a_{km}^{\dagger} a_{lm} \tag{2}$$

are the generators of the SU(3) group. This model describes M identical particles in three, M-fold degenerate, single-particle levels ε_i. Like the authors of Refs. [3,4] we assume $\varepsilon_2=-\varepsilon_0=\varepsilon=1$, $\varepsilon_1=0$, a vanishing interaction for particles in the same level and an equal interaction for particles in different levels.

The classical Hamiltonian $H_{\rm cl}$, given by

$$H_{\rm cl}=\lim_{M\to\infty} \left\langle \mathrm{SU}(3) \left| \frac{H}{M} \right| \mathrm{SU}(3) \right\rangle , \tag{3}$$

where $|\mathrm{SU}(3)\rangle$ is the coherent state, has been discussed in great detail in [3,4] and may be written as

$$H_{\rm cl}=-1+\tfrac{1}{2}q_1^2(1-\chi)+\tfrac{1}{2}q_2^2(2-\chi)+\tfrac{1}{2}p_1^2(1+\chi)+\tfrac{1}{2}p_2^2(2+\chi) + \tfrac{1}{4}\chi[(q_1^2+q_2^2)^2-(p_1^2+p_2^2)^2-(q_1^2-p_1^2)(q_2^2-p_2^2)-4q_1q_2p_1p_2] , \tag{4}$$

where $\chi=MV/\varepsilon$. The phase space Ω is a compact hypersphere with the equation $q_1^2+q_2^2+p_1^2+p_2^2 \le 2$.

In order to analyze the stability of the system, we calculated the periodic orbits of this model using Hamilton's equations of (4):

$$\dot{\mathbf{z}}=J\nabla H_{\rm cl}(\mathbf{z},\chi) , \tag{5}$$

where

$$\mathbf{z}=(q_1,q_2,p_1,p_2) ,$$

$$\nabla=\left[\frac{\partial}{\partial q_1},\frac{\partial}{\partial q_2},\frac{\partial}{\partial p_1},\frac{\partial}{\partial p_2}\right] ,$$

and J is the 4×4 symplectic matrix:

$$J=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} , \tag{6}$$

where I is the 2×2 identity matrix.

If $\mathbf{w}$ is a vector of the tangent space $T\Omega_{\mathbf{z}}$ of the phase-space manifold Ω at $\mathbf{z}$, its time evolution is given by

$$\dot{\mathbf{w}}=J\frac{\partial^2 H(\mathbf{z})}{\partial \mathbf{z}^2}\mathbf{w} . \tag{7}$$

By (5) and (7) the Lyapunov exponents can be calculated [5]:

$$\lambda(\mathbf{z})=\lim_{t\to\infty} \frac{1}{t}\ln|\mathbf{w}(t)| . \tag{8}$$

In terms of the stability matrix $M(0,t)$, defined in the usual way,

$$M_{ij}(0,t)=\frac{\partial z_i(t)}{\partial z_j(0)} , \tag{9}$$

1063-651X/93/47(6)/4556(4)/$06.00

$\lambda(\mathbf{z})$ can be written as

$$\lambda(\mathbf{z}) = \lim_{t\to\infty} \frac{1}{t} \ln|M(0,t)| , \quad (10)$$

where $|M(0,t)|$ is the norm of the matrix $M(0,t)$. This matrix can be calculated by solving its equations of motion:

$$\dot{M} = J \frac{\partial^2 H(\mathbf{z})}{\partial \mathbf{z}^2} M , \quad (11)$$

with the initial conditions

$$M(0) = I , \quad (12)$$

where I is the 4×4 identity matrix. The calculation of the Lyapunov exponents is related to that of the eigenvalues σ_i of the matrix $M(0,T)$:

$$\lambda_i(\mathbf{z}) = \frac{1}{T} \ln\sigma_i . \quad (13)$$

Now, using the unitary nature of M, a periodic orbit is unstable if

$$\mathrm{Tr}(M) > 4 \quad \text{or} \quad \mathrm{Tr}(M) < 0 \quad (14)$$

and stable if

$$0 < \mathrm{Tr}(M) < 4 . \quad (15)$$

It is also interesting to study the change of stability of periodic trajectories as a function of the coupling constant χ. In Fig. 1 the ratio between the number of stable orbits and the number of total orbits with period $T<30$ is plotted versus χ. For the coupling constant $\chi \in (0,3] T_{\min} \simeq 3$, as shown in Ref. [6]. As can be seen, the sensitivity of the orbits to a small change of χ is quite different for different values of χ, reflecting the transition order$\to$chaos as the coupling constant increases. Incidentally, the results shown in Fig. 1 are the generalization of those discussed in [6,7]. In fact, in the papers mentioned above, we limited ourselves to three different families of short periodic orbits with the initial condition along the axis q_1, q_2 and near the static potential minima.

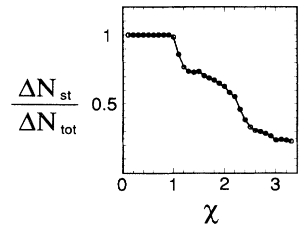

FIG. 1. Ratio between the number of stable periodic orbits and the number of total periodic orbits vs χ, with $T<30$; $T_{\min} \simeq 3$.

In order to apply the quantal criteria to our system, the eigenvalues of Hamiltonian (1) must be calculated. A natural basis can be written $|bc\rangle$, meaning b particles in the second level, c in the third, and, of course, $M-b-c$ in the first level; in this way $|00\rangle$ is the ground state with all the particles in the lowest level. We can write the general basis state as

$$|bc\rangle = \left[\frac{1}{b!c!}\right]^{1/2} G_{21}^{b} G_{31}^{c} |00\rangle , \quad (16)$$

where $\sqrt{1/b!c!}$ is the normalizing constant.

From the commutation relation of G_{kl} we can calculate expectation values of H/M and thus, eigenvalues and eigenstates of H/M; in this way the energy spectrum range is independent of the number of particles:

$$\langle b'c'|\frac{H}{M}|bc\rangle = \frac{1}{M}(-M+b+2c)\delta_{bb'}\delta_{cc'} - \frac{\chi}{2M^2} Q_{b'c',bc} , \quad (17)$$

where

$$\begin{aligned} Q_{b'c',bc} = {} & \sqrt{b(b-1)(M-b-c+1)(M-b-c+2)}\,\delta_{b-2,b'}\delta_{cc'} \\ & + \sqrt{(b+1)(b+2)(M-b-c)(M-b-c-1)}\,\delta_{b+2,b'}\delta_{cc'} \\ & + \sqrt{c(c-1)(M-b-c+1)(M-b-c+2)}\,\delta_{b,b'}\delta_{c-2,c'} \\ & + \sqrt{(c+1)(c+2)(M-b-c)(M-b-c-1)}\,\delta_{b,b'}\delta_{c+2,c'} \\ & + \sqrt{(b+1)(b+2)c(c-1)}\,\delta_{b+2,b'}\delta_{c-2,c'} + \sqrt{b(b-1)(c+1)(c+2)}\,\delta_{b-2,b'}\delta_{c+2,c'} . \end{aligned}$$

The expectation values $\langle H/M\rangle$ are real and symmetric. For any given number of particles M, we can set up the complete basis state, calculate the matrix elements of $\langle H/M\rangle$, and then diagonalize $\langle H/M\rangle$ to find its eigenvalues. $\langle H/M\rangle$ connects states with $\Delta b=-2,0,2$ and $\Delta c=-2,0,2$ only, which simplifies matters. States with b,c even; b,c odd; b even and c odd; b odd and c even are grouped together. Thus $\langle H/M\rangle$ becomes block diagonal, containing four blocks that can be diagonalized separately; these matrices are referred to as *ee*, *oo*, *oe*, and *eo*.

When the parameter $\chi=0$ the Hamiltonian consists of two oscillators and there are many degeneracies, but for $\chi\neq0$ these degeneracies are obviously broken.

For a large number of particles (semiclassical limit), we calculated the density of quasicrossings outside the degeneracy region as a function of the parameter χ:

$$\rho(\chi) = \frac{\Delta N}{\Delta\chi} , \quad (18)$$

where ΔN is the number of quasicrossings in the parameter range $\Delta\chi=0.01$. To obtain ΔN we fixed three values $\chi-\Delta\chi$, χ, and $\chi+\Delta\chi$ and imposed that

$$s_i(\chi-\Delta\chi)>s_i(\chi) , \quad (19)$$

$$s_i(\chi+\Delta\chi)>s_i(\chi) , \quad (20)$$

where $s_i(\chi)=E_{i+1}(\chi)-E_i(\chi)$.

The results (Fig. 2) show a maximum of quasicrossing for $\chi=2$ for all classes, in agreement with the transition to chaos of Fig. 1.

In order to study the sensitivity of energy levels to small changes of the parameter χ we used the statistics $\Delta^2(E)$, defined in the usual way [8]:

$$\Delta^2(E_i)=|E_i(\chi+\Delta\chi)+E_i(\chi-\Delta\chi)-2E_i(\chi)| , \quad (21)$$

which measures the curvature of E_i in a small range $\Delta\chi$. To remove the secular variation of the level density, each spectrum was mapped into one which has a constant level density by a numerical procedure described in Ref. [9]. Figure 3 shows $\Delta^2(E)$ for different values of χ; we note that the maximum value of $\Delta^2(E)$ corresponds to the $\chi=2$ value.

In conclusion, in the study of the transition from order to chaos, there is, in agreement with the authors of [10,11], a good correspondence between the classical approach, based on the stability matrix and Lyapunov exponents, and the quantal one, based on the quasicrossing distribution and the Δ^2 statistics.

For the sake of completeness in Fig. 4 the distribution $P(S)$ of spacings S between adjacent levels for the *eo* class (nearest-neighbor spacing distribution) has been calculated and compared to the Brody distribution [12,13]:

$$P(S)=\alpha(q+1)S^q\exp(-\alpha S^{q+1}) , \quad (22)$$

with

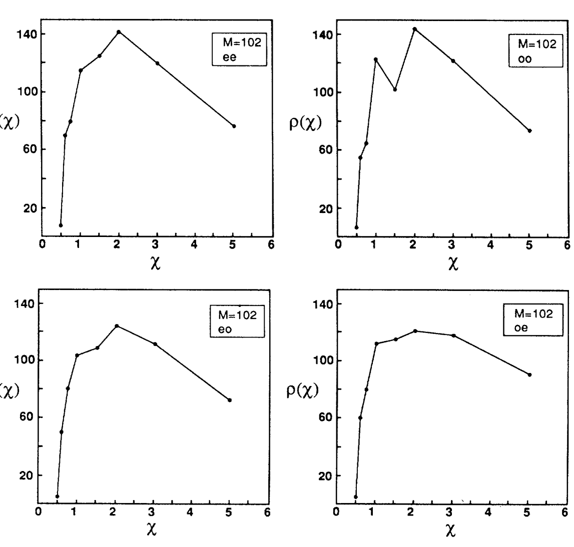

FIG. 2. Density of quasicrossings vs χ for all classes (*e* denotes even and *o* denotes odd).

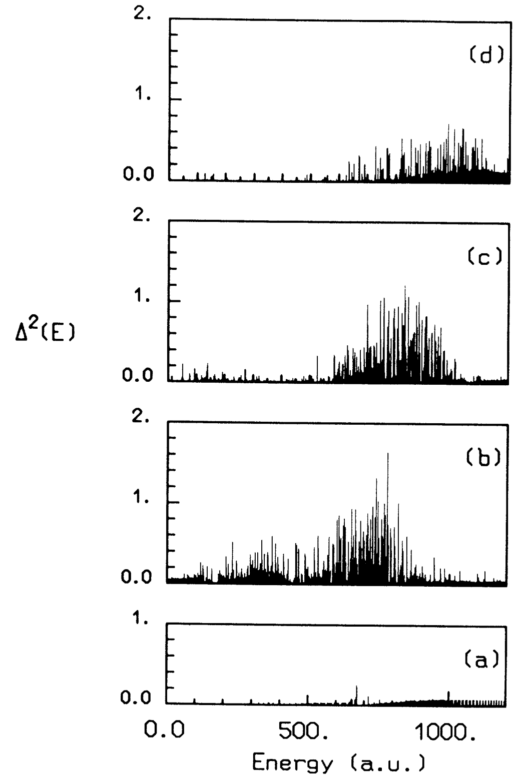

FIG. 3. $\Delta^2(E)$ vs E for different values of χ for the *eo* class; $M=102$; (a) $\chi=0.5$, (b) $\chi=2$, (c) $\chi=3$, (d) $\chi=5$.

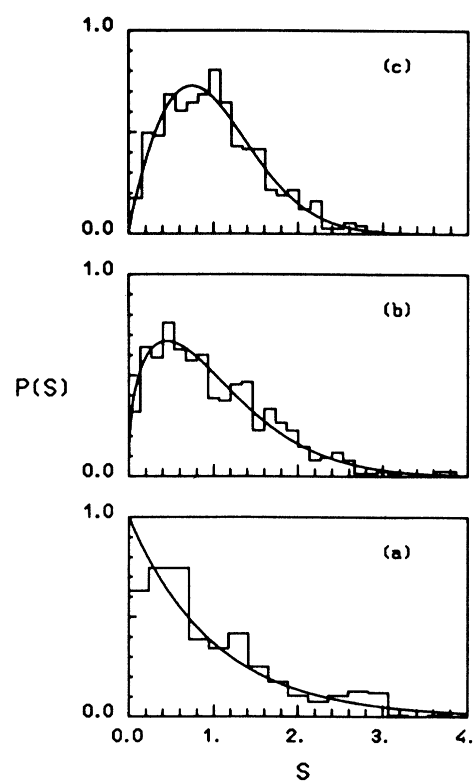

FIG. 4. $P(S)$ vs S for different values of χ for the *eo* class; $M=102$; (a) $\chi=0.75$, (b) $\chi=2$, (c) $\chi=3$.

$$\alpha = \left[\Gamma \left[\frac{q+2}{q+1} \right] \right]^{q+1}, \quad 0 \le q \le 1 . \tag{23}$$

The distribution (22) interpolates between the Poisson distribution ($q=0$) and the Wigner distribution ($q=1$). As can be seen from Fig. 4 this statistic also confirms the smooth transition from the regular to the chaotic regime discussed in the paper.

The authors are greatly indebted to Mr. G. Salmaso for his valuable computational assistance. This work has been partially supported by the Ministero dell'Università e della Ricerca Scientifica e Tecnologica (MURST).

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